Grassmanian Permutations

Fix $0 \leq k \leq n$.

$$w = w_1 < w_2 < \cdots < w_k \,,\ w_{k+1} < \cdots < w_n \in S_n$$

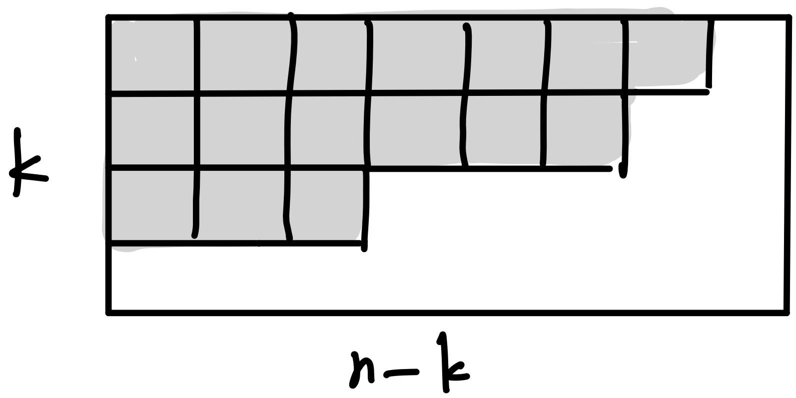
such permutations are called Grassmanian

can be identified with partitions λ

$$\lambda = (\lambda_1, \ldots, \lambda_k)$$

that fit inside $k \times (n-k)$ rectangle.

i.e. $\quad n-k \geq \lambda_1 \geq \cdots \geq \lambda_k \geq 0$



$$w(\lambda) = \lambda_k + 1 < \lambda_{k-1} + 2 < \lambda_{k-2} + 3 < \cdots < \lambda_1 + k$$

and all others in increasing order.

<u>Thm</u> For Grassmanian perm $w(\lambda)$ $\lambda \subseteq k\times(n-k)$

$$\mathfrak{S}_w(x_1,\dots,x_n) = s_\lambda(x_1,\dots,x_k)$$

Basics for S_n

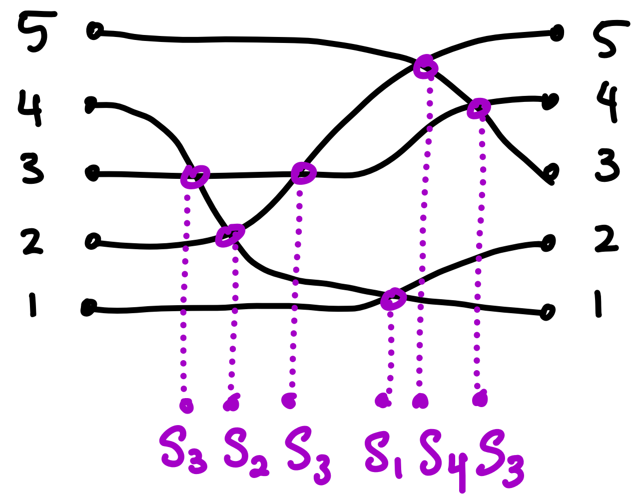
<u>Thm</u> S_n is generated by $s_1, s_2, \dots, s_{n-1}$ with rels

$$\left.\begin{array}{l} (1)\ s_i^2 = 1 \\ (2)\ s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \\ (3)\ s_i s_j = s_j s_i \text{ for } |i-j| \geq 2 \end{array}\right\} \text{Coxeter Relations}$$

Wiring Diagrams of Permutations

<u>Ex</u> $w = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 5 & 4 & 1 & 3 \end{pmatrix}$

assume the diagram is sufficiently generic.



then $s_3 s_2 s_3 s_1 s_4 s_3 = w^{-1}$

$s_3 s_4 s_1 s_3 s_2 s_3 = w$

<u>Def</u> the length $\ell(w)$ of $w \in S_n$

$$\ell(w) = \min(\ell \text{ s.t. } w = s_{i_1} s_{i_2} \cdots s_{i_\ell})$$

$$= \min(\#\text{ crossings in wiring diagram})$$

$$= \#\text{ inversions in } w$$

$\uparrow$ $\mathrm{inv}(w) = \#\{i<j : w_i > w_j\}$

<u>Def</u> A <u>reduced</u> decompositin $w = s_{i_1} \cdots s_{i_\ell}$ is a decomposition of w of length $\ell(w)$

Key Lemma Any two reduced decompositions of the same permutation can be obtained from each other by sequence of moves (2) & (3)

Divided Difference Operators

Recall $\partial_i : f \longrightarrow \frac{1}{x_i - x_{i+1}}(1-s_i)f$

Lemma: $\partial_1, \dots, \partial_{n-1}$ satisfy the relations

$$\left.\begin{array}{ll} (1)' & \partial_i^2 = 0 \\ (2)' & \partial_i \partial_{i+1} \partial_i = \partial_{i+1} \partial_i \partial_{i+1} \\ (3)' & \partial_i \partial_j = \partial_j \partial_i \quad \text{if } |i-j| \geq 2 \end{array}\right\} \text{nil Coxeter Relations}$$

Define $\partial_w = \partial_{i_1} \partial_{i_2} \cdots \partial_{i_\ell}$ for $w = s_{i_1} s_{i_2} \cdots s_{i_\ell}$

is reduced decomposition.

well defined by Key Lemma

Def $\mathfrak{S}_w = \partial_{w^{-1}w_0}(x^\delta)$ is Schubert Polynomial.

This shows well definedness.

Proposition:

$$\partial_{w_0} = \frac{1}{\prod_{i<j}(x_i - x_j)} \left(\sum_{w \in S_n} (-1)^{\ell(w)} w \right)$$

Cor $s_\lambda(x_1, \dots, x_n) = \partial_{w_0}(x^{\lambda+\delta})$